

Adjacency preserving mappings on real symmetric matrices

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Abstract

Let S_n denote the space of all $n \times n$ real symmetric matrices. For $n \geq 2$ we characterize maps $\Phi : S_n \rightarrow S_m$, which preserve adjacency, i.e. if $A, B \in S_n$ and $\text{rank}(A - B) = 1$, then $\text{rank}(\Phi(A) - \Phi(B)) = 1$.

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Introduction

Wen-Ling Huang and Peter Šemrl in [1] characterized adjacency preserving maps from H_n to H_m , where H_n denotes the $n \times n$ hermitian matrices over \mathbb{C} . They improved the results going back to Hua ([2], [3]). See also [11]-[19], [20]-[25]. This article considers adjacency preserving mappings from S_n to S_m , where S_n denotes the $n \times n$ symmetric matrices over \mathbb{R} . The authors of [1] suggested this problem in their article. It turns out that the ideas and methods of their paper work in the real case as well (with modifications in some places).

The proof of the complex case uses results by Wen-Ling Huang, Roland Höfer and Zhe-Xian Wan [4], which hold in the real case as well. We also take advantage of a theorem of Alexandrov [5] on Minkowski geometries. Alternatively, we can use the recent result [7] of Wen-Ling Huang on adjacency preserving maps from S_2 to S_2 . (This paper is also based on projective geometry.)

The main result of this paper is **Theorem 1.4**.

1 Notation

We will consider only matrices over \mathbb{R} . Let $M_n = M_n(\mathbb{R})$ be the space of all $n \times n$ matrices. Let S_n denote the linear subspace of all symmetric matrices in M_n , i.e. all $A \in M_n$ such that $A = A^T$, where A^T is the transpose of A . Let $GL(n)$ denote the group of all invertible matrices in M_n . Let $\text{lin } Z$ denote the real linear span of a set Z (in some vector space). We will often look at matrices in M_n as linear operators on \mathbb{R}^n . So for $A \in M_n$, $\text{Im } A = A\mathbb{R}^n$ is the *image* of A or the *column space* of A .

If we consider $x, y \in \mathbb{R}^n$ as $n \times 1$ matrices, $xy^T = x \otimes y$ is the rank one matrix with the property $(x \otimes y)z = \langle z, y \rangle x$ for $z \in \mathbb{R}^n$.

If $P \in S_n$ and $P^2 = P = P^T \neq 0$, then we call P a *projection*, as it is the orthogonal projection on $\text{Im } P$. Two projections P, Q are orthogonal, $P \perp Q$, iff $PQ = 0$. If x is a unit vector, then $x \otimes x$ is the projection on $\text{lin } \{x\}$.

Let e_1, \dots, e_n be the standard basis in \mathbb{R}^n and let $e_i \otimes e_j = E_{ij}$ be the matrix unit, i.e. the matrix with 1 in place (i, j) and zeros elsewhere.

We know that for $R, T \in M_n$, $\text{Im } (R + T) \subseteq \text{Im } R + \text{Im } T$ and so $\text{rank } (R + T) \leq \text{rank } R + \text{rank } T$.

For $A, B \in S_n$ let $d(A, B) = \text{rank } (A - B)$. Then (S_n, d) is a metric space. We will often use

Lemma 1.1 *Let $A, B, C \in M_n$ and $A + B = C$. Then $\text{rank } A = \text{rank } B + \text{rank } C$ iff $\text{Im } A = \text{Im } B \oplus \text{Im } C$.*

Two matrices A, B are *adjacent* if $d(A, B) = 1$, i.e. $\text{rank}(A - B) = 1$. If $d(A, B) = k$, there is a sequence of consecutively adjacent matrices $A_0 = A, A_1, \dots, A_k = B$ (see Proposition 5.5 in [8]). Conversely, if there is such a sequence, it is straightforward that $d(A, B) \leq k$.

Let $A, B \in S_n$ be adjacent. The line $l(A, B)$ joining A and B is the set consisting of A, B and all $Y \in S_n$, which are adjacent to both A and B . By [8],

$$l(A, B) = \{A + \lambda(B - A); \lambda \in \mathbb{R}\}.$$

If $P \in S_n$ is a projection, let $PS_nP = \{PAP; A \in S_n\} = \{C \in S_n; PCP = C\}$.

Proposition 1.2 *For $A, B, S \in S_n$, $R \in GL(n)$, and $c \in \mathbb{R} \setminus \{0\}$ we have $d(A + S, B + S) = d(A, B) = d(RAR^T, RBR^T) = d(cA, cB)$. Consequently, these are equivalent:*

- i) A is adjacent to B ;
- ii) $A + S$ is adjacent to $B + S$;
- iii) RAR^T is adjacent to RBR^T ;
- iv) cA is adjacent to cB .

Corollary 1.3 *Let $\Phi : S_n \rightarrow S_m$ be a map preserving adjacency, i.e. A is adjacent to B implies $\Phi(A)$ is adjacent to $\Phi(B)$. Let $\Psi(A) = \Phi(A) - \Phi(0)$ for $A \in S_n$. Then Ψ is adjacency preserving and $\Psi(0) = 0$.*

Theorem 1.4 (MAIN THEOREM) *Let m, n be natural numbers, $n \geq 2$. Let $\Phi : S_n \rightarrow S_m$ be a map preserving adjacency, with $\Phi(0) = 0$. Then either:*

- i) *There is a rank one matrix $B \in S_m$ and a function $f : S_n \rightarrow \mathbb{R}$ such that for $A \in S_n$*

$$\Phi(A) = f(A)B.$$

*In this case we say Φ is a **degenerate** adjacency preserving map.*

- ii) *We have $c \in \{-1, 1\}$, $R \in GL(m)$ such that for $A \in S_n$,*

$$\Phi(A) = cR \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} R^T.$$

*In this case we say Φ is a **standard** map. (Obviously, in this case $m \geq n$.)*

2 Preliminary results

We borrow Lemma 2.1. in [4]:

Lemma 2.1 *Let $G \in S_n$ and let l be a line in S_n . Then either:*

- i) *There is k such that $d(G, X) = k$ for all $X \in l$ or*
- ii) *There is a point $K \in l$ such that $d(G, X) = d(G, K) + 1$ for all $X \in l$, $X \neq K$.*

Lemma 2.2 *Let $A \in S_n$ be adjacent to both R and λR , where $R \in S_n$ has rank one and $\lambda \neq 1$. Then $A = \mu R$ for some $\mu \in \mathbb{R}$, $\mu \neq 1, \lambda$.*

Proof: Since $\lambda \neq 1$, R and λR are adjacent and A is contained in the line $l(R, \lambda R)$. So $A = R + \mu'(R - \lambda R) = \mu R$ and $\mu \neq \lambda, 1$.

□

The following lemma is slightly more general then Lemma 2.3. in [1].

Lemma 2.3 *Let $P \in M_n$ be an idempotent and $A, B \in M_n$ such that $P = A + B$ and rank $P = \text{rank } A + \text{rank } B$. Then A, B are idempotents and $AB = BA = 0$.*

Proof: By Lemma 1.1, $\text{Im } P = \text{Im } A \oplus \text{Im } B$. So if $Px = 0$, $Ax = Bx = 0$ and thus $\ker P \subset \ker A$. For $y \in \text{Im } A \subset \text{Im } P$, $Py = y = Ay + By$, hence $y - Ay = By$. Since $y - Ay \in \text{Im } A$, we have $By = 0$. Thus $BA = 0$ and $A^2 = A$. By symmetry, $AB = 0$ and $B^2 = B$.

□

Lemma 2.4 *Let $P_1, P_2, \dots, P_k \in S_n$ be mutually orthogonal rank one projections and $P = P_1 + \dots + P_k$. Let $\xi(1), \dots, \xi(n)$ be an orthonormal system in \mathbb{R}^n such that $P_i(\xi(i)) = \xi(i)$ for $i = 1, \dots, k$. Then $P_i(\xi(j)) = \delta_{ij}\xi(j)$. Let V be the orthogonal matrix defined by $Ve_i = \xi(i)$ for $i = 1, \dots, n$, so that $\xi(i)$ is the i -th column of V . Then $V^T P_i V = E_{ii}$ for $i = 1, \dots, k$. If $A \in PS_n P = \{C \in S_n | PCP = P\}$, then*

$$V^T A V = \begin{bmatrix} q(A) & 0 \\ 0 & 0 \end{bmatrix}$$

where $q(A) \in S_k$. We have $q(P_i) = E_{ii}$ for $i = 1, \dots, k$ and $q(P) = E_{11} + \dots + E_{kk}$.

The mapping $q : PS_nP \rightarrow S_k$ is linear, bijective, and $q(AB) = q(A)q(B)$ for $A, B, AB \in PS_nP$. So $q(A^2) = q(A)^2$ and q is a Jordan isomorphism. It preserves the distance d and thus adjacency. Also $q(ABA) = q(A)q(B)q(A)$ for all $A, B \in PS_nP$. All these properties are shared by the mappings $h : S_k \rightarrow S_n$ and $q^{-1} : S_k \rightarrow PS_nP$, where

$$h(B) = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$$

and $q^{-1}(B) = Vh(B)V^T$.

Lemma 2.5 Let k, n be natural numbers with $3 \leq k \leq n$. Let $\lambda_1, \dots, \lambda_k$ be nonzero real numbers and $P_1, \dots, P_k \in S_n$ mutually orthogonal rank one projections. Let $A = \sum_{j=1}^k \lambda_j P_j$. Let $B \in S_n$ have rank $B = \text{rank } A = k$ and let B be adjacent to $A - \lambda_i P_i$ for all i . Assume that $d(B, \lambda_i P_i) = k - 1$ for all i . Then $B = A$.

Proof: By Lemma 1.1, $\text{Im } B = \text{Im } (\lambda_i P_i) \oplus \text{Im } (B - \lambda_i P_i)$. So $\text{Im } P_i \subset \text{Im } B$ for all i . If $P = P_1 + \dots + P_k$, then $\text{Im } P \subset \text{Im } B$ and $\text{rank } P = k$, so $\text{Im } P = \text{Im } B$ and consequently $PB = B = BP$. Thus $A, B \in PS_nP$. Using notation from Lemma 2.4, $q(A), q(B) \in S_k$ and $q(B)$ is adjacent to $q(A) - \lambda_i q(P_i)$, $d(q(B), \lambda_i q(P_i)) = k - 1$. Also, $q(P) = E_{11} + \dots + E_{kk} = I_k$ and $q(A), q(B)$ have maximal rank as elements in S_k . Thus we may assume that $k = n$ and A, B are invertible in S_n , $P_1 + \dots + P_n = I$.

Now $1 = \text{rank } (B - A + \lambda_i P_i) = \text{rank } (A^{-1}B - I + \lambda_i A^{-1}P_i)$. But $A^{-1} = \sum \lambda_i^{-1} P_i$, so $\lambda_i A^{-1}P_i = P_i$. Let $C = B^{-1}A \in M_n$. Then $C \in GL(n)$ and $1 = \text{rank } (C^{-1} - (I - P_i)) = \text{rank } (I - C(I - P_i))$.

Now $I = (I - C(I - P_i)) + C(I - P_i)$ and $\text{rank } C(I - P_i) = \text{rank } (I - P_i) = n - 1$. By Lemma 2.2, $C(I - P_i)$ is an idempotent.

Let f_1, \dots, f_n be an orthonormal basis of \mathbb{R}^n such that $P_i f_i = f_i$. Then for $j \neq i$, $(I - P_i)f_j = f_j$, so $Cf_j = C(I - P_i)f_j = C(I - P_i)C(I - P_i)f_j = C(I - P_i)Cf_j$. Since C is invertible,

$$(I - P_i)Cf_j = f_j \text{ for } j \neq i.$$

Let $Cf_j = \sum_{m=1}^n a_m f_m$. Then $(I - P_i)Cf_j = Cf_j - P_i Cf_j = Cf_j - a_i f_i = \sum_{m \neq i} a_m f_m = f_j$. So $a_m = 0$ for $m \neq i, j$ and $a_j = 1$. Thus $Cf_j = f_j + a_i f_i$. Since $n \geq 3$, there exists k , $1 \leq k \leq n$, $k \neq i, j$. So $Cf_j = f_j + a_k f_k$ also. Thus $Cf_j = f_j$ for all j and $C = I$. This implies $A = B$.

□

Lemma 2.6 Let $A, B \in S_m$ and let $\text{rank } A = 1$. If $\text{rank } (A + \lambda B) = 1$ for every $\lambda \in \mathbb{R}$, then $B = 0$.

Proof: If $\text{rank } B \geq 2$, then there exists a nonsingular 2×2 submatrix in B . For $\lambda \neq 0$, we have $\text{rank } (A + \lambda B) = \text{rank } (B + \frac{1}{\lambda}A) \geq 2$ for λ large enough, since the chosen submatrix of $(B + \frac{1}{\lambda}A)$ will be nonsingular. Therefore, $\text{rank } B \leq 1$.

If $B \neq 0$, then B is adjacent to 0. Also, $A + B$ is adjacent to 0 and B , so $A + B \in l(B, 0)$. Thus $A + B = \mu B$ and $A + (1 - \mu)B = 0$ – a contradiction.

□

Lemma 2.7 *Let $A, B \in S_n$ have rank n ($n \geq 2$), with $A \neq B$. There exists a natural number k and invertible matrices $A = A_0, A_1, \dots, A_k = B$ such that the neighbours in this sequence are adjacent and there is a matrix $C_j \in l(A_j, A_{j+1})$ with $\text{rank } C_j = n - 1$ for $j = 0, \dots, k - 1$.*

Proof: This is a consequence of Lemmas 2.5 and 2.6 in [4] and is stated in the proof of Lemma 3.1 in the same paper.

Lemma 2.8 *Let $\Phi : S_n \rightarrow S_m$ be an adjacency preserving map. Let $A, B \in S_n$ be adjacent. Then $\Phi(l(A, B)) \subset l(\Phi(A), \Phi(B))$. The restriction of Φ to $l(A, B)$ is injective.*

Proof: If $\lambda_1 \neq \lambda_2$ and $C_i = A + \lambda_i(B - A) \in l(A, B)$ ($i = 1, 2$), then C_1 is adjacent to C_2 and therefore $\Phi(C_1)$ is adjacent to $\Phi(C_2)$, thus $\Phi(C_1) \neq \Phi(C_2)$.

□

Lemma 2.9 *Let $\Phi : S_n \rightarrow S_m$ ($n \geq 2$) be a map preserving adjacency and $\Phi(0) = 0$. Let $\max\{\text{rank } \Phi(A) | A \in GL(n)\} = k$. If $k \geq 2$ and for every singular $A \in S_n$ we have $\text{rank } \Phi(A) < k$, then $\text{rank } \Phi(B) = k$ for every invertible $B \in S_n$.*

Proof: Let $A, B \in S_n \cap GL(n)$ with $A \neq B$ and let $\text{rank } \Phi(A) = k$. By Lemma 2.7, there exists a natural number r and invertible matrices $A = A_0, A_1, \dots, A_r = B$ such that the neighbours in this sequence are adjacent and for $j = 0, \dots, r - 1$ there is a matrix $C_j \in l(A_j, A_{j+1})$ with $\text{rank } C_j = n - 1$. Hence $\text{rank } \Phi(C_j) < k$. Now $\text{rank } \Phi(A) = k$, $\text{rank } \Phi(A_1) \leq k$, $\text{rank } \Phi(C_0) < k$. Lemma 2.1 (for $G = 0$) tells us that $\Phi(C_0)$ is the only point on the line $l(\Phi(A), \Phi(A_1))$ with rank less than k . Since $C_0 \neq A_1$, Lemma 2.8 tells us that $\Phi(C_0) \neq \Phi(A_1)$. So $\text{rank } \Phi(A_1) = k$. Proceeding in this way we find $\text{rank } \Phi(A_j) = k$ for all j , so $\text{rank } \Phi(B) = k$.

□

Lemma 2.10 *Let $\Phi : S_n \rightarrow S_m$ ($n \geq 2$) be a map preserving adjacency. If there are $A, B \in S_n$ with $d(\Phi(A), \Phi(B)) = n$, then $d(\Phi(X), \Phi(Y)) = d(X, Y)$ for all $X, Y \in S_n$ and Φ is injective.*

Proof: For $n = m$ this was proved (in even greater generality) by Wen-ling Huang (Corollary 3.1 in [23]).

We know that $d(X, Y) = k \geq 1$ implies the existence of a sequence $X = X_0, X_1, \dots, X_k = Y$ of consecutively adjacent matrices. If $\Psi : S_n \rightarrow S_m$ is adjacency preserving, the neighbours in the sequence $\Psi(X_0), \Psi(X_1), \dots, \Psi(X_k)$ are also adjacent and therefore $d(\Psi(X), \Psi(Y)) \leq k$. So

$$d(\Psi(X), \Psi(Y)) \leq d(X, Y)$$

for any adjacency preserving map Ψ .

Now the map Ψ , defined by $\Psi(X) = \Phi(X + A) - \Phi(A)$ for $X \in S_n$ is adjacency preserving by Proposition 1.2 and $\Psi(0) = 0$. We note that $\text{rank}(\Psi(B - A)) = d(\Phi(B), \Phi(A)) = n$.

If $Z \in S_n$ is singular, $\text{rank}(\Psi(Z)) = d(\Psi(Z), \Psi(0)) \leq d(Z, 0) = \text{rank } Z \leq n - 1$. Lemma 2.9 tells us that $\text{rank}(\Psi(X)) = n$ for every $X \in S_n \cap GL(n)$. In particular, if $d(C, A) = n$, i.e. $\text{rank}(C - A) = n$, then $n = \text{rank}(\Psi(C - A)) = \text{rank}(\Phi(C) - \Phi(A)) = d(\Phi(C), \Phi(A))$.

Let $X, Y \in S_n$ be such that $d(X, Y) = n$. For λ large enough, $d(\lambda I, A) = \text{rank}(\lambda I - A) = n$ and $d(\lambda I, X) = n$. If we set $C = \lambda I$ above, we see $d(\Phi(\lambda I), \Phi(A)) = n$. We may substitute λI for A , A for B in the previous argument and get $d(\Phi(\lambda I), \Phi(X)) = n$. Repeating this procedure we get $d(\Phi(X), \Phi(Y)) = n$.

We have proven that $d(X, Y) = n$ implies $d(\Phi(X), \Phi(Y)) = n$. Suppose now $d(Z, W) = \text{rank}(Z - W) = k < n$, with $k \geq 1$. There is U orthogonal such that $Z - W = U(\lambda_1 E_{11} + \dots + \lambda_k E_{kk})U^T$, with $\lambda_1, \dots, \lambda_k$ nonzero. Let $G = W - U(E_{k+1, k+1} + \dots + E_{nn})U^T$. Then $d(G, W) = \text{rank}(G - W) = n - k$ and $(Z - W) + (W - G) = Z - G$ is invertible. Since Φ does not increase the metric d , $n = d(Z, G) = d(Z, W) + d(W, G) \geq d(\Phi(Z), \Phi(W)) + d(\Phi(W), \Phi(G)) \geq d(\Phi(Z), \Phi(G)) = n$. So $d(Z, W) = d(\Phi(Z), \Phi(W))$.

If $\Phi(X) = \Phi(Y)$ and $X \neq Y$, then $d(X, Y) \geq 1$, so $d(\Phi(X), \Phi(Y)) \geq 1$ – a contradiction.

□

Lemma 2.11 *Let $m > n \geq 2$ and let $A_1, B_1 \in S_n$ with $A_1 \neq B_1$. If $A, B \in S_m$ are such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and C is adjacent to both A and B , then there is $C_1 \in S_n$ such that

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof: The matrices $A - C$ and $C - B$ have rank one. So $A - B = (A - C) + (C - B)$ has rank one or two. If A is adjacent to B , then C lies on the line $l(A, B)$, so $C = A + \lambda(B - A)$ has the desired form.

If $A - B$ has rank two, then $\text{Im}(A - B) = \text{Im}(A - C) \oplus \text{Im}(C - B)$ by Lemma 1.1. So $\text{Im}(A - C) \subset \text{Im}(A - B)$ and $C = A - (A - C)$ has the desired form.

□

Lemma 2.12 *Let $m > n \geq 2$ and let $\Phi : S_n \rightarrow S_m$ be an adjacency preserving map with $\Phi(0) = 0$. Let*

$$\Phi(I) = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}$$

where $K \in S_n$ has rank n . Then for all $A \in S_n$,

$$\Phi(A) = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $A_1 \in S_n$.

Proof: Since $n = d(\Phi(I), \Phi(0))$, Lemma 2.10 tells us that d preserves the distance. Suppose $P \in S_n$ is a projection of rank one. Then $d(0, P) = 1$, $d(I, P) = n - 1$, so $d(\Phi(I), \Phi(P)) = n - 1$ and $d(0, \Phi(P)) = 1$. Thus

$$n = \text{rank } \Phi(I) = \text{rank } \Phi(P) + \text{rank } (\Phi(I) - \Phi(P)).$$

By Lemma 1.1, $\text{Im } \Phi(I) = \text{Im } \Phi(P) \oplus \text{Im } (\Phi(I) - \Phi(P))$, so $\text{Im } \Phi(P) \subset \text{Im } \Phi(I)$ and $\Phi(P)$ has the desired form.

If $A = \lambda P$, then A lies on the line $l(0, P)$, so $\Phi(A)$ lies on the line $l(0, \Phi(P))$, so $\Phi(A) = \mu \Phi(P)$ has the desired form.

Now we use the induction on the rank of A . Suppose we have proved the lemma for all matrices of rank $k \geq 1$. Let $\text{rank } A = k + 1$. There is U orthogonal and nonzero numbers $\lambda_1, \dots, \lambda_{k+1}$ such that $A = U(\lambda_1 E_{11} + \dots + \lambda_{k+1} E_{k+1, k+1})U^T$. The matrix A is adjacent to $B = U(\lambda_2 E_{22} + \dots + \lambda_{k+1} E_{k+1, k+1})U^T$ and to $C = U(\lambda_1 E_{11} + \dots + \lambda_k E_{kk})U^T$. So $\Phi(A)$ is adjacent to

$$\Phi(B) = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Phi(C) = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $B_1, C_1 \in S_n$ and $B_1 \neq C_1$. By Lemma 2.10, $\Phi(B) \neq \Phi(C)$. We use Lemma 2.11.

□

3 Adjacent matrices in S_2

Wen-Ling Huang proved the following result (Corollary 2 in [7]):

Let $\Phi : S_2 \rightarrow S_2$ be an adjacency preserving map. Suppose there are $A, B \in S_2$ such that $\Phi(A)$ and $\Phi(B)$ are not adjacent. Then there are $c \in \{-1, 1\}$ and $T \in GL(2)$, $S \in S_2$ such that $\Phi(X) = cTXX^T + S$ for $X \in S_2$.

This Corollary implies the main result of this section, Proposition 3.5. But we can also proceed in a way analogous to that in [1].

Lemma 3.1 *Let $\Phi : S_2 \rightarrow S_2$ be a map such that A is adjacent to B iff $\Phi(A)$ is adjacent to $\Phi(B)$. Then Φ is injective.*

Proof: If there are $A, B \in S_2$ such that $d(\Phi(A), \Phi(B)) = 2$, then, by Lemma 2.10, Φ is injective.

Suppose now that $d(\Phi(X), \Phi(Y)) \leq 1$ for all $X, Y \in S_2$. We will show this is impossible. Since E_{11} and E_{22} are not adjacent, $\Phi(E_{11})$ and $\Phi(E_{22})$ are not adjacent. Therefore $\Phi(E_{11}) = \Phi(E_{22})$. Similarly, $\Phi(2E_{11}) = \Phi(E_{22})$. On the other hand, E_{11} is adjacent to $2E_{11}$, so $\Phi(E_{11})$ is adjacent to $\Phi(2E_{11}) = \Phi(E_{11})$ – a contradiction.

□

We denote by Q the quadratic form on \mathbb{R}^n , defined by $Q(x) = x_n^2 - x_1^2 - x_2^2 - \dots - x_{n-1}^2$. Then $Q(x - y)$ is the **Lorentz separation** of x and y . A bijective linear transformation $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a **Lorentz transformation** if $Q(Lx) = Q(x)$ for all $x \in \mathbb{R}^n$. All Lorentz transformations on \mathbb{R}^n form the **Lorentz group**. A mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a **Weyl transformation** if there are: $\alpha \in \mathbb{R} \setminus \{0\}$, a Lorentz transformation L and $b \in \mathbb{R}^n$ such that $f(x) = \alpha Lx + b$ for all $x \in \mathbb{R}^n$.

The following theorem is due to Alexandrov [5]. We quote it from Lester [6] p. 929, who rediscovered it.

Theorem 3.2 *Let D be an open connected subset of \mathbb{R}^n and let $f : D \rightarrow \mathbb{R}^n$ be an injective mapping such that $Q(x - y) = 0$ iff $Q(f(x) - f(y)) = 0$. Then f is the restriction of conformal mapping.*

Any conformal mapping on \mathbb{R}^n is a Weyl transformation (see [6], p. 929 or [10], pp. 132-133) and that is all we will need:

Corollary 3.3 *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an injective mapping such that $Q(x - y) = 0$ iff $Q(f(x) - f(y)) = 0$. Then f is a Weyl transformation.*

We have the linear bijection $T : \mathbb{R}^3 \rightarrow S_2$, defined by

$$Tx = \begin{bmatrix} x_3 + x_1 & x_2 \\ x_2 & x_3 - x_1 \end{bmatrix}.$$

Now $\det(Tx - Ty) = \det(T(x - y)) = Q(x - y)$. Therefore:

$$Tx \text{ is adjacent to } Ty \text{ iff } x \neq y \text{ and } Q(x - y) = 0. \quad (1)$$

The following is taken from the book [9] on Hyperbolic Geometry by Ramsey and Richtmyer, pp. 246-250. If $L = [l_{ij}] \in M_3$ is a Lorentz matrix, then $|\det L| = 1$ and $|l_{33}| \geq 1$. If $\det L = 1$ and $l_{33} > 1$, then L is a **restricted Lorentz matrix**. If L is a restricted Lorentz matrix, then there is a matrix $P_1 \in M_2$ with $\det P_1 = 1$ such that

$$T(Lx) = P_1(Tx)P_1^T$$

for all $x \in \mathbb{R}^3$. Now $K = -E_{11} + E_{22} + E_{33} = K^{-1}$ is a Lorentz matrix with $\det K = -1$. For $Q = E_{12} + E_{21} \in S_2$ we have $T(Kx) = Q(Tx)Q^T$.

If $L \in M_3$ is any Lorentz matrix, then there is $r \in \{-1, 1\}$ such that rL or LK or rLK is a restricted Lorentz matrix. It follows that for any Lorentz matrix $L \in M_3$ we have

$$T(Lx) = c_1 P(Tx)P^T \quad (2)$$

where $c_1 \in \{-1, 1\}$, $|\det P| = 1$ and $x \in \mathbb{R}^3$.

Corollary 3.4 *Let $\Phi : S_2 \rightarrow S_2$ be a map such that A is adjacent to B iff $\Phi(A)$ is adjacent to $\Phi(B)$. Then there exist $c \in \{-1, 1\}$, $R \in GL(2)$ and $S \in S_2$ such that*

$$\Phi(A) = cRAR^T + S \quad (A \in S_2).$$

Proof: We consider the mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, defined by

$$f(x) = T^{-1}\Phi(Tx).$$

By Lemma 3.1, f is injective. If $x \neq y$ and $Q(x - y) = 0$, then Tx is adjacent to Ty , so $\Phi(Tx)$ is adjacent to $\Phi(Ty)$, so $Q(f(x) - f(y)) = 0$. If $f(x) = f(y)$, then $x = y$.

If $f(x) \neq f(y)$ and $Q(f(x) - f(y)) = 0$, then $\Phi(Tx)$ is adjacent to $\Phi(Ty)$ by (1), so Tx is adjacent to Ty and $Q(x - y) = 0$.

We see that $Q(x - y) = 0$ iff $Q(f(x) - f(y)) = 0$. By Corollary 3.3, there exist $\alpha \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}^3$ and a Lorentz matrix $L \in GL(3)$ such that $f(x) = \alpha Lx + b$ for all $x \in \mathbb{R}^3$, hence

$$\Phi(Tx) = \alpha T(Lx) + Tb.$$

By (2), there are $c_1 \in \{-1, 1\}$ and $P \in GL(n)$ such that

$$\Phi(Tx) = \alpha c_1 P(Tx)P^T + Tb,$$

i.e.

$$\Phi(A) = cRAR^T + S$$

for $A \in S_2$, where $c \in \{-1, 1\}$, $R \in GL(2)$ and $S \in S_2$.

□

Proposition 3.5 *Let $\Phi : S_2 \rightarrow S_2$ be an adjacency preserving mapping. Suppose $d(\Phi(G), \Phi(H)) = 2$ for some $G, H \in S_2$. Then there are $c \in \{-1, 1\}$, $R \in GL(2)$ and $S \in S_2$ such that*

$$\Phi(A) = cRAR^T + S.$$

Proof: By Lemma 2.10, $d(\Phi(X), \Phi(Y)) = d(X, Y)$ for all $X, Y \in S_2$. So $\Phi(X)$ is adjacent to $\Phi(Y)$ iff X is adjacent to Y . We use Corollary 3.4.

□

4 Proof of theorem 1.4

Let $n \geq 2$ and let $\Phi : S_n \rightarrow S_m$ be a mapping preserving adjacency, $\Phi(0) = 0$. Theorem 1.4 states that Φ is either degenerate or a standard map.

Lemma 4.1 *Theorem 1.4 is true if $n = 2$.*

Proof: If $m = 1$, Φ is a degenerate map. Let $m \geq 2$. We consider two cases.

Case 1: Let $d(\Phi(A), \Phi(B)) \leq 1$ for all A, B .

Then $\text{rank } \Phi(A) \leq 1$ for all A . Since E_{11} is adjacent to 0, $\Phi(E_{11})$ is adjacent to $\Phi(0) = 0$, so $\text{rank } \Phi(E_{11}) = 1$. Let $A \in S_2$. Then $d(\Phi(A), \Phi(E_{11})) \leq 1$. So $\Phi(A) = \Phi(E_{11})$ or $\Phi(A)$ is adjacent to $\Phi(E_{11})$. In the latter case, if $\Phi(A) \neq 0$, then $\Phi(A)$ is adjacent to 0, so $\Phi(A) \in l(0, \Phi(E_{11}))$, thus $\Phi(A) = \lambda\Phi(E_{11})$. So $\Phi(A) = \lambda\Phi(E_{11})$ in any case. Thus Φ is a degenerate map.

Case 2: We have $A, B \in S_2$ such that $d(\Phi(A), \Phi(B)) = 2$.

If $m = 2$, then Proposition 3.5 ends the proof. Let $m > 2$. By Lemma 2.10, Φ preserves the distance and is injective. So $d(\Phi(I), 0) = 2 = \text{rank } \Phi(I)$. Since $\Phi(I) \in S_m$, there is $U \in M_m$ orthogonal such that

$$U\Phi(I)U^T = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Let $\Psi(A) = U\Phi(A)U^T$ for $A \in S_2$. Then Ψ is distance preserving and $\Psi(0) = 0$. By Lemma 2.12,

$$\Psi(A) = \begin{bmatrix} \Psi_1(A) & 0 \\ 0 & 0 \end{bmatrix}$$

where $\Psi_1(A) \in S_2$ and $\Psi_1(0) = 0$.

Obviously, $d(\Psi(A), \Psi(B)) = d(\Psi_1(A), \Psi_1(B))$. So $\Psi_1 : S_2 \rightarrow S_2$ is distance preserving. By Proposition 3.5, there are $c \in \{-1, 1\}$ and $R \in GL(2)$ such that $\Psi_1(A) = cRAR^T$. Let

$$W = \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} \in GL(m).$$

Then

$$\Psi(A) = cW \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} W^T$$

and

$$\Phi(A) = cU^T W \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} (U^T W)^T.$$

□

Lemma 4.2 *Let $n \geq 2$ and let $\Phi : S_n \rightarrow S_m$ be a map preserving adjacency, with $\Phi(0) = 0$. Let*

$$\Phi(I) = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \in S_m$$

where $I_n \in M_n$ is the identity matrix. Then we can find $U \in M_n$ orthogonal such that for all $A \in S_n$ we have

$$\Phi(A) = \begin{bmatrix} UAU^T & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof: Obviously $m \geq n$. If $m > n$, then by Lemma 2.12, for all $A \in S_n$ we have

$$\Phi(A) = \begin{bmatrix} \Phi_1(A) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Phi_1(A) : S_n \rightarrow S_n$ and $\Phi_1(I) = I$. Also $\Phi_1(0) = 0$ and Φ_1 preserves adjacency. Thus it suffices to prove the theorem for $m = n$. We will use induction on n . We know our Lemma is true for $n = 2$ using Proposition 3.5. Suppose it is valid for $n - 1$, where $n \geq 3$.

Let $P \in S_n$ be a projection with rank $P = k$. Since $d(\Phi(I), \Phi(0)) = n$, Lemma 2.10 says $d(\Phi(A), \Phi(B)) = d(A, B)$ for all $A, B \in S_n$. So rank $\Phi(A) =$

rank A for all A and $\text{rank } \Phi(P) = k$. Since $d(I, P) = n - k$, we have $d(I, \Phi(P)) = n - k$, so $I = \Phi(P) + R_1$, where $\text{rank } R_1 = n - k$. By Lemma 2.3, $\Phi(P) = Q$ is a projection. So Φ maps projections into projections of the same rank.

Suppose $k = n - 1$. By Lemma 2.4, there is a Jordan isomorphism $q : PS_nP \rightarrow S_{n-1}$ which preserves the distance d .

Since Q is similar to $E_{11} + \dots + E_{n-1, n-1}$, there is $W \in M_n$ orthogonal such that

$$WQW^T = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \in M_n.$$

We define $f_1 : S_{n-1} \rightarrow S_n$ by $f_1(B) = W\Phi(q^{-1}(B))W^T$. Then

$$f_1(I_{n-1}) = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \in S_n.$$

We use the induction hypothesis. There is $U_1 \in M_{n-1}$ orthogonal such that for $B \in S_{n-1}$

$$f_1(B) = \begin{bmatrix} U_1 B U_1^T & 0 \\ 0 & 0 \end{bmatrix} \in S_n.$$

For $A \in PS_nP$ we have

$$\Phi(A) = W^T \begin{bmatrix} U_1 q(A) U_1^T & 0 \\ 0 & 0 \end{bmatrix} W \in M_n.$$

The mapping Φ restricted to PS_nP is a Jordan isomorphism (in particular linear) and if $AB = 0$, then $\Phi(A)\Phi(B) = 0$. Thus Φ maps projections in PS_nP into projections of the same rank and preserves the orthogonality of projections in PS_nP .

Since $n \geq 3$, for any rank one projections $P_1, P_2 \in S_n$ with $P_1 P_2 = 0$ there is a projection P of rank $n - 1$ such that $P_1, P_2 \in PS_nP$. So $\Phi_1(P_1), \Phi_1(P_2)$ are rank one projections and $\Phi(P_1)\Phi(P_2) = 0$.

Thus $\Phi(E_{11}), \dots, \Phi(E_{nn})$ are mutually orthogonal rank one projections. By Lemma 2.4, there is $V \in M_n$ orthogonal such that $V\Phi(E_{ii})V^T = E_{ii}$ ($i = 1, \dots, n$). By exchanging Φ with the map $A \mapsto V\Phi(A)V^T$ we may assume $\Phi(E_{ii}) = E_{ii}$ for $i = 1, \dots, n$.

Let $j \neq i$ and $R = E_{ii} + E_{jj}$. Since $n \geq 3$, there is a projection P of rank $n - 1$ such that $RS_nR \subset PS_nP$. By the preceding paragraph $\Phi(R) = \Phi(E_{ii}) + \Phi(E_{jj}) = E_{ii} + E_{jj} = R$ and for $A \in RS_nR$ we have $\Phi(A) = \Phi(R)\Phi(A)\Phi(R) = R\Phi(A)R$, so $\Phi(A) \in RS_nR$. Also, Φ restricted to RS_nR is linear, injective, preserving the products (if the products are in RS_nR).

By Lemma 2.4, we have the Jordan isomorphism $q : RS_nR \rightarrow S_2$, such that $q(E_{ii}) = E_{11}$ and $q(E_{jj}) = E_{22}$. The map $K = q\Phi q^{-1} : S_2 \rightarrow S_2$ is

adjacency preserving, $K(0) = 0$, $K(I_2) = I_2$. By the induction hypothesis, there is $U_2 \in M_2$ orthogonal such that $K(B) = U_2 B U_2^T$ for $B \in S_2$. Since $K(E_{11}) = E_{11}$, $K(E_{22}) = E_{22}$, $U_2 = \text{diag}(\lambda_1, \lambda_2)$, with $\lambda_1, \lambda_2 \in \{-1, 1\}$. It follows there is $w_{ij} \in \{-1, 1\}$ such that $K(E_{12} + E_{21}) = w_{ij}(E_{12} + E_{21})$, hence $\Phi(E_{ij} + E_{ji}) = w_{ij}(E_{ij} + E_{ji})$ and $\Phi(\alpha E_{ii} + \beta(E_{ij} + E_{ji}) + \gamma E_{jj}) = \alpha E_{ii} + w_{ij}\beta(E_{ij} + E_{ji}) + \gamma E_{jj}$ for $\alpha, \beta, \gamma \in \mathbb{R}$. Let $w_{ii} = 1$ for all i .

For $b \in \mathbb{R}$ let $B = E_{ii} + b(E_{ij} + E_{ji}) + b^2 E_{jj} \in S_n$. Then B has rank one and $B \in R S_n R$. If $A = [a_{ij}] \in S_n$ has rank one, then there exists a projection Q with $\text{rank } Q = n - 1$ such that $A, B \in Q S_n Q$. Since Φ restricted to $Q S_n Q$ is a Jordan map, $\Phi(BAB) = \Phi(B)\Phi(A)\Phi(B)$. Also $B = BR = RB$ and so $BAB = B(RAR)B$. But $RAR = a_{ii}E_{ii} + a_{ij}(E_{ij} + E_{ji}) + a_{jj}E_{jj}$. We know that $\Phi(B) = E_{ii} + w_{ij}b(E_{ij} + E_{ji}) + b^2 E_{jj}$, so $R\Phi(B) = \Phi(B)R$ and $\Phi(B(RAR)B) = \Phi(BAB) = \Phi(B)\Phi(A)\Phi(B) = (\Phi(B)R)\Phi(A)(R\Phi(B))$. So

$$\Phi(B(RAR)B) = \Phi(B)(R\Phi(A)R)\Phi(B) \quad (b \in \mathbb{R}). \quad (3)$$

If $\Phi(A) = [a'_{ij}]$, $R\Phi(A)R = a'_{ii}E_{ii} + a'_{ij}(E_{ij} + E_{ji}) + a'_{jj}E_{jj}$. Equation 3 implies $a'_{ii} = a_{ii}$, $a'_{jj} = a_{jj}$ and $a'_{ij} = w_{ij}a_{ij}$. So for all i, j

$$a'_{ij} = w_{ij}a_{ij}. \quad (4)$$

Suppose now $T \in S_n$ is such that $t_{ij} = 1$ for all i, j . Then T has rank one and consequently $\Phi(T) = [w_{ij}] \in S_n$ has rank one. There exists $\lambda \in \mathbb{R}$ such that $\Phi(T) = \lambda Q_2$, where Q_2 is a rank one projection. There exists a unit vector $x \in \mathbb{R}^n$ such that $Q_2 = x \otimes x$. Now $1 = w_{11} = \langle \Phi(T)e_1, e_1 \rangle = \lambda \langle Q_2 e_1, e_1 \rangle = \lambda \|Q_2 e_1\|^2$. So $\lambda > 0$. Therefore, if $y = x\sqrt{\lambda}$, then $\Phi(T) = y \otimes y = y^T y$, so $w_{ij} = y_i y_j$ for all i, j . But $w_{ii} = y_i^2 = 1$, so $y_i \in \{-1, 1\}$ for all i . Therefore, if $V_2 = \text{diag}(y_1, \dots, y_n)$, $V_2^T = V_2$ is orthogonal: $V_2^2 = I$ and $\Phi(A) = V_2 A V_2$. Thus $V_2 \Phi(A) V_2 = A$ for all $A \in S_n$ with $\text{rank } A = 1$. By exchanging Φ with the map $A \rightarrow V_2 \Phi(A) V_2^T$ we may assume $\Phi(A) = A$ for all $A \in S_n$ with $\text{rank } A = 1$.

Suppose $B \in S_n$ has rank less than n . Then $B = \sum_{i=1}^{n-1} \lambda_i P_i$, where $\lambda_i \in \mathbb{R}$ and P_i are mutually orthogonal rank one projections. Let $P = \sum_{i=1}^{n-1} P_i$. Since Φ , restricted to $P S_n P$ is linear,

$$\Phi(B) = \sum_{i=1}^{n-1} \Phi(\lambda_i P_i) = \sum_{i=1}^{n-1} \lambda_i P_i = B.$$

Let $C \in S_n$ be invertible. Again, $C = \sum_{i=1}^n \alpha_i Q_i$, where $\alpha_i \neq 0$ and Q_i are mutually orthogonal rank one projections. Let $G = C - \alpha_i Q_i$. Then $\text{rank } G = n - 1$. Also G is adjacent to C , so the same is true for $\Phi(C)$ and $\Phi(G) = G = C - \alpha_i Q_i$. Since Φ preserves the distance, we have $n - 1 =$

$\text{rank } G = d(C, \alpha_i P_i) = d(\Phi(C), \alpha_i Q_i)$. Since $\text{rank } \Phi(C) = \text{rank } C = n$, Lemma 2.5 implies $\Phi(C) = C$.

□

Lemma 4.3 *Let $\Phi : S_n \rightarrow S_m$ ($m, n \geq 3$) be an adjacency preserving map and $\Phi(0) = 0$. Suppose that for every projection $P \in S_n$ with $\text{rank } P = n - 1$ there is a rank one projection Q such that $\Phi(PS_nP) \subset \mathbb{R}Q$. Then Φ is a degenerate adjacency preserving map.*

Proof: Let $P, P_1 \in S_n$ be projections of rank $n - 1$. Let $\Phi(PS_nP) \subset \mathbb{R}Q$ and $\Phi(P_1S_nP_1) \subset \mathbb{R}Q_1$, where Q, Q_1 are rank one projections. There is a projection R of rank one such that $R \in PS_nP \cap P_1S_nP_1$. Since R is adjacent to 0, $\Phi(R)$ is adjacent to 0, so $\Phi(R) = \lambda Q = \mu Q_1$ with $\lambda, \mu \neq 0$. Thus $Q = Q_1$ and $\Phi(B) \in \mathbb{R}Q$ for all $B \in S_n$ with $\text{rank } B \leq n - 1$. There is an orthogonal matrix V such that $VQV^T = E_{11}$. Exchanging Φ for the map $X \mapsto V\Phi(X)V^T$ we may assume $Q = E_{11}$. So $\Phi(B) \in \mathbb{R}E_{11}$ for all B with $\text{rank } B \leq n - 1$.

If $A \in S_n$ is invertible, then

$$A = \sum_{j=1}^n \lambda_j P_j, \quad (5)$$

where λ_j are nonzero and P_j are mutually orthogonal rank one projections. So A is adjacent to $B = \sum_{j=2}^n \lambda_j P_j$. Thus $\Phi(A)$ is adjacent to $\Phi(B) = \lambda E_{11}$ and $\text{rank } \Phi(A) \leq 2$.

Case 1: Assume $\text{rank } \Phi(A) \leq 1$ for all $A \in S_n \cap GL(n)$.

We claim $\Phi(S_n \cap GL(n)) \subset \mathbb{R}E_{11}$. Suppose, on the contrary, that there exists A invertible such that $\Phi(A) = Z \notin \mathbb{R}E_{11}$. Then $\text{rank } Z = 1$. Let $A = \sum_{j=1}^n \lambda_j P_j$ as in (5). Let $B = \sum_{j=2}^n \lambda_j P_j$. Then Z is adjacent to $\Phi(B) = \lambda E_{11}$ and to 0. If $\lambda \neq 0$, then Z lies on the line $l(0, \lambda E_{11})$ – a contradiction. So $\Phi(B) = 0$. By Lemma 2.8, Φ maps the line $l(B, A) = \{B + \lambda P_1; \lambda \in \mathbb{R}\}$ into the line $l(0, Z) = \mathbb{R}Z$ injectively. So there is $\lambda \in \mathbb{R}$, $\lambda \neq \lambda_1$ such that $\Phi(B + \lambda P_1) = \frac{1}{2}Z$. Now $C_1 = \sum_{j=2}^{n-1} \lambda_j P_j + \lambda_1 P_1$ is adjacent to A and $C_2 = \sum_{j=2}^{n-1} \lambda_j P_j + \lambda P_1$ is adjacent to $B + \lambda P_1$, so $B_1 = \Phi(C_1)$ is adjacent to Z , $B_2 = \Phi(C_2)$ is adjacent to $\frac{1}{2}Z$. Both B_1 and B_2 are in $\mathbb{R}E_{11}$. Since Z is adjacent to $\frac{1}{2}Z$, we have $B_1, B_2 \in l(Z, \frac{1}{2}Z) \subset \mathbb{R}Z$. So $B_1, B_2 = 0$ – a contradiction with the fact that B_1, B_2 are adjacent. Thus, if $\text{rank } \Phi(A) \leq 1$ for all invertible $A \in S_n$, the proof is finished.

Case 2: Suppose there is $A \in S_n \cap GL(n)$ with $\text{rank } \Phi(A) = 2$.

By Lemma 2.9, $\text{rank } \Phi(X) = 2$ for every $X \in S_n \cap GL(n)$. So $\Phi(I)$ and $\Phi(I + E_{11})$ have rank two and are adjacent. Let $D = E_{22} + \dots + E_{nn}$. Then D

is adjacent to I and to $I + E_{11}$, implying that $\Phi(D)$ is adjacent to $\Phi(I)$. Thus $\Phi(D) \neq 0$ and $\Phi(D) = \lambda E_{11}$ for some $\lambda \neq 0$. Since $\Phi(D)$ is adjacent to $\Phi(I)$ and to $\Phi(I + E_{11})$, we have $\Phi(I) = \lambda E_{11} + K$ and $\Phi(I + E_{11}) = \lambda E_{11} + K'$, with $\text{rank } K = \text{rank } K' = 1$. We claim that

$$\text{rank } (K + \mu E_{11}) = \text{rank } (K' + \mu E_{11}) = 2 \quad \text{for } \mu \neq 0. \quad (6)$$

In fact, if $\mu \neq 0$ and $\text{rank } (K + \mu E_{11}) = 1$, then $K + \mu E_{11}$ is adjacent to 0 and to μE_{11} , so $K + \mu E_{11}$ is on the line $l(0, \mu E_{11})$, so $K + \mu E_{11} = \gamma \mu E_{11}$ – a contradiction to the fact that $\text{rank } \Phi(I) = 2$.

Now $\Phi(E_{11} + \sum_{j=3}^n E_{jj}) = \lambda' E_{11}$ and is adjacent to $\Phi(I) = \lambda E_{11} + K$. Thus $\text{rank } ((\lambda - \lambda')E_{11} + K) = 1$, which implies by (6) that $\lambda = \lambda'$. Also $\Phi(2E_{11} + \sum_{j=3}^n E_{jj}) = \lambda'' E_{11}$ is adjacent to $\Phi(I + E_{11}) = \lambda E_{11} + K'$. As before, $\lambda'' = \lambda$. But $E_{11} + \sum_{j=3}^n E_{jj}$ is adjacent to $2E_{11} + \sum_{j=3}^n E_{jj}$, so λE_{11} is adjacent to λE_{11} – a contradiction.

So $\text{rank } \Phi(A) \leq 1$ for all invertible $A \in S_n$ and the proof is finished. \square

Lemma 4.4 *Let $\Phi : S_n \rightarrow S_m$ ($m, n \geq 3$) be an adjacency preserving map with $\Phi(0) = 0$. Assume that for every projection P with $\text{rank } P = n - 1$ the restriction of Φ to PS_nP is a standard map. Then Φ is a standard adjacency preserving map.*

Proof: Let $D = E_{11} + \dots + E_{n-1, n-1}$. There are $c \in \{-1, 1\}$ and $T \in GL(m)$ such that for $B \in DS_nD$,

$$\Phi(B) = cT \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} T^T.$$

If $\Psi(X) = cT^{-1}\Phi(X)(T^{-1})^T$ for $X \in S_n$, then $\Psi(0) = 0$, Ψ preserves adjacency and

$$\Psi(B) = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \quad (B \in DS_nD). \quad (7)$$

In particular, $\Psi(E_{11}) = E_{11}$. If Q is a rank one projection, then we claim $\Psi(Q) \geq 0$. There exists a projection P of rank $n - 1$, such that $E_{11}, Q \in PS_nP$. The restriction of Φ to PS_nP is a standard map. This is also true for the restriction of Ψ to PS_nP . A standard map Ω has either the property $X \geq 0$ implies $\Omega(X) \geq 0$ or $X \geq 0$ implies $-\Omega(X) \geq 0$. Since $\Psi(E_{11}) \geq 0$, $\Psi(Q) \geq 0$.

Thus $\Psi(E_{nn}) \geq 0$ and $\Psi(E_{nn})$ is adjacent to 0, so $\text{rank } \Psi(E_{nn}) = 1$. So $\Psi(E_{nn}) = sx \otimes x$ for some unit vector $x \in \mathbb{R}^m$ and $s > 0$. We show that $x \notin$

$\text{lin } \{e_1, \dots, e_{n-1}\}$ (This implies $m \geq n$). If $x = \alpha_1 e_1 + \dots + \alpha_{n-1} e_{n-1} \in \mathbb{R}^m$ and $y = \alpha_1 e_1 + \dots + \alpha_{n-1} e_{n-1} \in \mathbb{R}^n$, then by (7) we have $\Psi(sy \otimes y) = sx \otimes x$. There exists a projection P_1 of rank $n - 1$ such that $E_{nn}, sy \otimes y \in P_1 S_n P_1$. But $\Psi(E_{nn}) = \Psi(sy \otimes y)$. Since the restriction of Ψ to $PS_n P$ is standard and thus injective, this is a contradiction.

We construct an invertible $R \in M_m$ such that $Re_i = e_i$ for $i = 1, \dots, n - 1$ and $Rx = s^{-\frac{1}{2}} e_n$. Now $Rs(x \otimes x)R^T = sRx \otimes Rx = e_n \otimes e_n = E_{nn}$. For $i, j \leq n - 1$ we have $RE_{ij}R^T = R(e_i \otimes e_j)R^T = (Re_i) \otimes (Re_j) = e_i \otimes e_j = E_{ij}$. We define $\Phi_1 : S_n \rightarrow S_m$ by $\Phi_1(X) = R\Psi(X)R^T$. Then (7) is true if we replace Ψ by Φ_1 . Also $\Phi_1(E_{nn}) = E_{nn}$.

Let $R_i = I - E_{ii} \in S_n$. Then $R_n = D$ and $\Phi_1(R_n) = E_{11} + \dots + E_{n-1, n-1} = E - E_{nn}$, where $E = E_{11} + \dots + E_{nn} \in S_m$. The restriction of Φ_1 to $R_i S_n R_i$ is a standard map and thus linear. So $\Phi_1(R_i) = \Phi_1(E_{11}) + \dots + \Phi_1(E_{i-1, i-1}) + \Phi_1(E_{i+1, i+1}) + \dots + \Phi_1(E_{nn}) = E - E_{ii}$ for $i = 1, \dots, n$.

Since I is adjacent to R_i , $\Phi_1(I)$ is adjacent to $E - E_{ii}$ for all i . Thus

$$E - E_{ii} = \Phi_1(I) + T_i, \text{ with rank } T_i = 1. \quad (8)$$

Thus $\text{rank } \Phi_1(I) \geq n - 2$. If $\text{rank } \Phi_1(I) = n - 2$, then $\text{rank } (E - E_{ii}) = \text{rank } \Phi_1(I) + \text{rank } T_i$, so $\text{Im } (E - E_{ii}) = \text{Im } \Phi_1(I) \oplus \text{Im } T_i$ by Lemma 1.1 and $\text{Im } \Phi_1(I)$ is a subspace in $\text{lin } (\{e_1, \dots, e_n\} \setminus \{e_i\})$ for all i . Thus $\text{Im } \Phi_1(I) = \{0\}$ and $\text{Im } (E - E_{ii}) = \text{Im } T_i$ - a contradiction, since $n \geq 3$.

Suppose $\text{rank } \Phi_1(I) = n - 1 = \text{rank } (E - E_{ii})$. By (8) we have $\Phi_1(I) = (E - E_{ii}) - T_i$ with $\text{rank } T_i = 1$. Let $T_i = \lambda_i(y_i \otimes y_i)$ with y_i a unit vector. If $y_i \notin \text{Im } (E - E_{ii})$, then $\text{rank } \Phi_1(I) = n$ and that is a contradiction. So $y_i \in \text{Im } (E - E_{ii})$ and $\text{Im } \Phi_1(I) \subset \text{Im } (E - E_{ii})$ for all i . Thus once again $\Phi_1(I) = \{0\}$ - a contradiction.

Thus $\text{rank } \Phi_1(I) = n$. Now $\Phi_1(I)$ is adjacent to $E - E_{ii}$ for all i . Also $n = d(\Phi_1(I), \Phi_1(0)) = \text{rank } \Phi_1(I)$. By Lemma 2.10, $d(I, E_{ii}) = n - 1 = d(\Phi_1(I), \Phi_1(E_{ii})) = d(\Phi_1(I), E_{ii})$. By Lemma 2.5, $\Phi_1(I) = E$.

By Lemma 4.2, we can find an orthogonal matrix $U \in M_n$ such that for $A \in S_n$

$$\Phi_1(A) = \begin{bmatrix} UAU^T & 0 \\ 0 & 0 \end{bmatrix}.$$

So Φ_1 is a standard map and therefore Φ is a standard map.

□

Lemma 4.5 *The statement of Theorem 1.4 is true for $n = 3$.*

Proof: Let $P \in S_3$ be any projection of rank 2. By 4.1, the mapping Φ restricted to PS_3P is either standard or degenerate. If Φ restricted to PS_3P

is degenerate for all projections $P \in S_3$ of rank 2, Lemma 4.3 tells us that Φ is degenerate. If Φ restricted to PS_3P is standard for all such P , then Lemma 4.4 tells us that Φ is a standard map.

Suppose there exist two projections P and Q of rank 2 such that Φ restricted to PS_3P is degenerate and Φ restricted to QS_3Q is standard. Then $m \geq 2$. If $R \in S_3$ has rank one, then R is adjacent to 0, so $\Phi(R)$ is adjacent to $\Phi(0) = 0$ and has rank one. There exists a rank one matrix $R_1 \in QS_3Q$ such that the rank one matrices $\Phi(R), \Phi(R_1)$ are linearly independent. (If this is not true, then $\Phi(R_1) = \lambda(R_1)\Phi(R)$ for all $R_1 \in QS_3Q$. Since Φ restricted to QS_3Q is standard and $\text{rank } Q = 2$, this is impossible.) There exists a rank two projection R_2 such that $R, R_1 \in R_2S_3R_2$. Then since $\Phi(R)$ and $\Phi(R_1)$ are linearly independent, Φ restricted to $R_2S_3R_2$ is not degenerate. Hence it is standard and therefore real linear. Thus for any rank one operator $R \in S_3$ we have

$$\Phi(\lambda R) = \lambda\Phi(R) \quad (\lambda \in \mathbb{R}). \quad (9)$$

Let $T \in S_3$. We define $\Phi_T : S_3 \rightarrow S_m$ by $\Phi_T(X) = \Phi(X + T) - \Phi(T)$. Then $\Phi_T(0) = 0$ and Φ_T is an adjacency preserving map by Proposition 1.2.

We show that Φ_T is neither standard nor degenerate. If Φ_T was standard, then Φ_T is real linear, so $\Phi(Y) = \Phi((Y - T) + T) = \Phi_T(Y - T) + \Phi(T) = \Phi_T(Y) - \Phi_T(T) + \Phi(T)$. Letting $Y = 0$ we get $0 = \Phi(0) = \Phi_T(0) + \Phi(T) - \Phi_T(T) = \Phi(T) - \Phi_T(T)$. So $\Phi = \Phi_T$ is standard – a contradiction.

If there exists a rank one operator G such that $\Phi_T(X) \in \mathbb{R}G$ for all $X \in S_3$, then for $Y \in S_3$ we have $\Phi(Y) = \Phi_T(Y - T) + \Phi(T) = \Phi(T) + \lambda(Y)G$. Thus $0 = \Phi(T) + \lambda(0)G$ and $\Phi(Y) = (\lambda(Y) - \lambda(0))G$ for all $Y \in S_3$ and Φ is degenerate – a contradiction.

As in the beginning of the proof of the Lemma, there are rank two projections P_T and Q_T such that the restriction of Φ_T to $P_TS_3P_T$ is degenerate and the restriction of Φ_T to $Q_TS_3Q_T$ is standard. If R is a rank one matrix in S_3 , then by (9) $\Phi_T(\lambda R) = \lambda\Phi_T(R)$ for $\lambda \in \mathbb{R}$. So $\Phi(\lambda R + T) - \Phi(T) = \Phi_T(\lambda R) = \lambda\Phi_T(R) = \lambda(\Phi(R + T) - \Phi(T))$, i.e.

$$\Phi(\lambda R + T) = \Phi(T) + \lambda(\Phi(R + T) - \Phi(T)). \quad (10)$$

Now we will prove that if $A_1, A_2, \dots, A_p \in S_3$ have rank one, then

$$\Phi(A_1 + A_2 + \dots + A_p) = \Phi(A_1) + \dots + \Phi(A_p)$$

by induction on p . It is true for $p = 1$. Assume it holds for p . Let $A_1, \dots, A_{p+1} \in S_3$ have rank one. Then

$$\Phi(A_1 + \dots + A_p + \lambda A_{p+1}) = \Phi(A_1 + \dots + A_p) + \lambda(\Phi(A_1 + \dots + A_{p+1}) - \Phi(A_1 + \dots + A_p))$$

by (10), so by the induction hypotesis,

$$\Phi(A_1 + \dots + A_p + \lambda A_{p+1}) = \Phi(A_1) + \dots + \Phi(A_p) + \lambda \Phi(A_1 + \dots + A_{p+1}) - \lambda(\Phi(A_1) + \dots + \Phi(A_p)).$$

Since $A_2 + \dots + A_p + \lambda A_{p+1}$ is adjacent to $A_1 + A_2 + \dots + A_p + \lambda A_{p+1}$, we have $\Phi(A_1 + \dots + A_p + \lambda A_{p+1})$ is adjacent to $(\Phi(A_2) + \dots + \Phi(A_p) + \lambda \Phi(A_{p+1}))$. Thus

$$\begin{aligned} \Phi(A_1 + \dots + A_p + \lambda A_{p+1}) &- \Phi(A_2) - \dots - \Phi(A_p) - \lambda \Phi(A_{p+1}) = \\ &= \Phi(A_1) + \lambda \Phi(A_1 + \dots + A_{p+1}) - \lambda \Phi(A_1) - \dots - \lambda \Phi(A_p) - \lambda \Phi(A_{p+1}) = \\ &= \Phi(A_1) + \lambda(\Phi(A_1 + \dots + A_{p+1}) - (\Phi(A_1) + \dots + \Phi(A_p) + \Phi(A_{p+1}))) \end{aligned}$$

has rank one for all $\lambda \in \mathbb{R}$. By Lemma 2.6,

$$\Phi(A_1 + \dots + A_p + A_{p+1}) = \Phi(A_1) + \dots + \Phi(A_{p+1}).$$

If $A \in S_3$, then $A = \sum_{i=1}^3 \lambda_i P_i$, where $P_i \in S_3$ are rank one projections. So $\Phi(A) = \sum_{i=1}^3 \lambda_i \Phi(P_i)$. It follows that Φ is linear.

Now Φ maps the rank two operator P into an operator of rank at most 1. By Lemma 2.10, $\text{rank } \Phi(A) \leq 2$ for all $A \in S_3$.

Let $\{f_1, f_2\} \subset \mathbb{R}^3$ be an orthonormal system such that $Qf_i = f_i$ for $i = 1, 2$. There exists $U \in M_3$ orthogonal such that $Ue_i = f_i$ for $i = 1, 2$. Then $QUe_i = Ue_i$ and $U^T Q U e_i = e_i$ for $i = 1, 2$. Since $U^T Q U \in S_3$ has rank two, we have $U^T Q U = E_{11} + E_{22} = E_2$. If $A \in E_2 S_3 E_2$, then $U^T Q U A U^T Q U = A$, so $Q(U A U^T)Q = U A U^T$, so $U A U^T \in Q S_3 Q$. Now Φ restricted to $Q S_3 Q$ is standard. So there are $c \in \{-1, 1\}$ and T invertible in M_m such that

$$\Phi(U A U^T) = c T \begin{bmatrix} U A U^T & 0 \\ 0 & 0 \end{bmatrix} T^T.$$

Therefore we may assume that for $A \in E_2 S_3 E_2$ we have

$$\Phi(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = h(A) \in S_m.$$

If F is any rank two projection in S_3 , the restriction of Φ to $F S_3 F$ is either standard or degenerate. (Look at the beginning of the proof of this Lemma.) The matrix $\Phi(E_{33})$ is adjacent to 0. Hence $\Phi(E_{33}) = s x \otimes x$, where $s \neq 0$ and x is a unit vector. If $x \notin \text{lin } \{e_1, e_2\}$, then $\Phi(I) = E_{11} + E_{22} + s x \otimes x$ has rank 3. But $\text{rank } \Phi(I) \leq 2$. So $x \in \text{lin } \{e_1, e_2\}$ and hence $\Phi(E_{33}) \in E_2 S_3 E_2$.

There exists a rank one projection $R_1 \in E_2 S_3 E_2$ such that

$$\Phi(R_1) = h(R_1) = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\Phi(E_{33})$ are linearly independent. There is a projection $R_2 \in S_3$ of rank 2 such that $R_1, E_{33} \in R_2 S_3 R_2$. The restriction of Φ to $R_2 S_3 R_2$ is standard, since $\Phi(R_1)$ and $\Phi(E_{33})$ are linearly independent.

Since $\Phi(R_1) = h(R_1) \geq 0$, $\Phi(E_{33}) \geq 0$. So

$$\Phi(E_{33}) = \begin{bmatrix} cP_2 & 0 \\ 0 & 0 \end{bmatrix}$$

where $c > 0$ and $P_2 \in S_2$ is a rank one projection. Let $U_1 \in M_2$ be an orthogonal matrix such that $U_1^T P_2 U_1 = E_{22}$. We define matrices $G \in M_3$, $V \in M_m$ by

$$G = \begin{bmatrix} U_1 & 0 \\ 0 & c^{-\frac{1}{2}} \end{bmatrix}, V = \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix}.$$

Then $GE_{33}G^T = c^{-1}E_{33}$, so $\Phi(GE_{33}G^T) = \begin{bmatrix} P_2 & 0 \\ 0 & 0 \end{bmatrix}$.

If we define $\Theta(X) = V^T \Phi(GXG^T)V$ for $X \in S_3$, then once more $\Theta : S_3 \rightarrow S_m$ is a linear adjacency preserving map with $\Theta(E_{33}) = E_{22}$. If $A \in E_2 S_3 E_2$, then $GAG^T \in E_2 S_3 E_2$, so

$$\Theta(A) = h(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

If P_1 is a rank two projection in S_3 , then, as before, Θ restricted to $P_1 S_3 P_1$ is either standard or degenerate. Now $\Theta(E_{22} + E_{33}) = 2E_{22}$, so Θ restricted to $(E_{22} + E_{33})S_2(E_{22} + E_{33})$ is degenerate. Therefore, $\Theta(E_{23} + E_{32}) = \alpha E_{22}$, with $\alpha \neq 0$.

Since $\Theta(E_{11} + E_{33}) = E_{11} + E_{22}$ has rank two, the restriction of Θ to $(E_{11} + E_{33})S_3(E_{11} + E_{33})$ is a standard map. As before, there are $c_1 \in \{-1, 1\}$ and $W_1 \in GL(m)$ such that for $A \in (E_{11} + E_{33})S_3(E_{11} + E_{33})$ we have

$$\Theta(A) = c_1 W_1 \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} W_1^T.$$

But $\Theta(E_{11}) = E_{11}$ and $\Theta(E_{33}) = E_{22}$. So

$$c_1 W_1 (e_1 \otimes e_1) W_1^T = c_1 (W_1 e_1) \otimes (W_1 e_1) = e_1 \otimes e_1.$$

This implies $c_1 = 1$ and $W e_1 = \pm e_1$. By exchanging W with $-W$ if necessary we may assume $W e_1 = e_1$. Similarly, $W e_3 = d e_2$, where $d \in \{-1, 1\}$. This implies

$$\Theta(E_{13} + E_{31}) = W(e_1 \otimes e_3 + e_3 \otimes e_1)W^T = W e_1 \otimes W e_3 + W e_3 \otimes W e_1 = d(E_{12} + E_{21}).$$

Let $A = [1, 1, 1]^T [1, 1, 1] = E_{11} + E_{22} + E_{33} + (E_{12} + E_{21}) + (E_{13} + E_{31}) + (E_{23} + E_{32})$. Since A has rank 1, A is adjacent to 0, so $\Theta(A)$ is adjacent to 0 and has rank one. We calculate $\Theta(A) = E_{11} + (2 + \alpha)E_{22} + (1 + d)(E_{12} + E_{21})$ and $\det \Theta(A) = 2 + \alpha - (1 + d)^2 = \alpha - 2d = 0$, since $d^2 = 1$. So $\Theta(E_{23} + E_{32}) = 2dE_{22}$.

Let now $B = [0, d, -1]^T [0, d, -1] = d^2E_{22} + E_{33} - d(E_{23} + E_{32})$. Then B has rank one and is adjacent to 0. But $\Theta(B) = (1 - d^2)E_{22} = 0$ – a contradiction.

□

End of proof of theorem 1.4

Let $n \geq 4$. Our induction hypothesis is that every adjacency preserving and zero preserving map from S_k to S_m ($2 \leq k < n$) is either standard or degenerate. Let $\Phi : S_n \rightarrow S_m$ be an adjacency preserving map and $\Phi(0) = 0$.

Let $P \in S_n$ be a projection of rank $n - 1$. By Lemma 2.4 we know that PS_nP is isomorphic to S_{n-1} . By the assumption, Φ restricted to PS_nP is either standard or degenerate. Let $Q \in S_n$ be another projection of rank $n - 1$. There exists a projection R of rank $n - 2 \geq 2$ with $PR = QR = R$, so that $R \in PS_nP \cap QS_nQ$. If Φ restricted to PS_nP is degenerate, then Φ restricted to RS_nR is degenerate, hence Φ restricted to QS_nQ cannot be standard and is thus degenerate. By Lemma 4.3, Φ is degenerate.

So if Φ restricted to PS_nP is degenerate, then Φ is degenerate. The remaining possibility is that Φ restricted to PS_nP is standard. Then obviously this is true if we replace P by Q . By Lemma 4.3, Φ is standard.

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